MATH 2B/5B Prep: Chain Rule

1. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\arctan\big(\sqrt{x}\big)$.

Solution: We have a composition of outer function $f(x) = \arctan(x)$ and inner function $g(x) = \sqrt{x}$. These have derivatives

$$f'(x) = \frac{1}{1+x^2} \qquad\qquad g'(x) = \frac{1}{2\sqrt{x}}$$

Then by chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan\big(\sqrt{x}\big) = \frac{\mathrm{d}}{\mathrm{d}x}f(g(x)) = f'(g(x))g'(x) = \frac{1}{1+(\sqrt{x})^2}\frac{1}{2\sqrt{x}} = \frac{1}{1+x}\frac{1}{2\sqrt{x}}$$

2. If $f(x) = \cos\big(e^{5x}\big)$ then what is $f'(x)$?

Solution: $f(x)$ is a composition of 3 functions, outer function $\cos(x)$, middle function e^x, and inner function $5x$. Their derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \qquad \frac{\mathrm{d}}{\mathrm{d}x}5x = 5$$

Using chain rule twice we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos\big(e^{5x}\big) = -\sin\big(e^{5x}\big)\frac{\mathrm{d}}{\mathrm{d}x}e^{5x} = -\sin\big(e^{5x}\big)e^{5x}5 = -5\sin\big(e^{5x}\big)e^{5x}$$

3. Compute the derivative of $\dfrac{1}{g(x)}$ in terms of $g(x)$ and $g'(x)$.

Solution: We consider $1/g(x)$ as the composition of outer function $f(x) = 1/x = x^{-1}$ with the inner function $g(x)$. Recall that $f'(x) = -1/x^2$. Then chain rule says

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)} = \frac{\mathrm{d}}{\mathrm{d}x}f(g(x)) = f'(g(x))g'(x) = -\frac{1}{g(x)^2}g'(x) = -\frac{g'(x)}{g(x)^2}$$